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ESTIMATE OF THE SINGULAR SET OF THE EVOLUTION PROBLEM FOR HARMONIC MAPS

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1. Introduction

Let $\mathcal{M}$, $\mathcal{N}$ be Riemannian manifolds of dimensions m, n $(m > 2)$ with metrics γ, g respectively. We consider the evolution of harmonic maps [3, (1.4)], [1,(1.6), (1.7)]:

$$\partial_t u - \Delta_{\mathcal{M}} u + \Gamma_{\mathcal{N}}(u)(\nabla u, \nabla u)_{\mathcal{M}} = 0, \qquad u|_{t=0} = u_0. \tag{1.1}$$

M. Struwe proved the following theorem.

[3, Theorem 6.1]. *Suppose $u: \mathbf{R}^m \times \mathbf{R}_+ \to \mathcal{N}$ is the limit of a sequence u_k of regular solutions to (1.1), with finite energy*

$$E(u_k(t)) \le E_0 < \infty, \quad \forall k \in \mathbf{N} \text{ and } t > 0$$

in the sense that $E(u(t)) \le E_0$ almost everywhere and that $\nabla u_k \to \nabla u$ weakly in $L^2(Q)$ for any compact $Q \subset \mathbf{R}^m \times \mathbf{R}_+$. Then u solves (1.1) in the classical sense and is regular on a dense open subset of $\mathbf{R}^m \times \mathbf{R}_+$ whose complement Σ has locally finite m-dimensional Hausdorff measure (with respect to the parabolic metric).

Here we give a better estimate on the singular set Σ.

Theorem. *If $t_0 > 0$, then $\Sigma \cap (\mathbf{R}^m \times \{t_0\})$ has finite $(m-2)$-dimensional Hausdorff measure.*

Remarks. In [1], with a general m-dimensional Riemannian manifold $\mathcal{M}$ replacing $\mathbf{R}^m$, Y. Chen and M. Struwe proved the *existence* of a solution to (1.1), which satisfies all the above conditions of [3, Theorem 6.1]. Here E_0 is the energy of the initial map $u(\cdot, 0)$.

In the case $m = 2$, M. Struwe [2] proved that Σ consists of at most finitely many points of $\mathcal{M} \times \mathbf{R}_+$.

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2. Notation

We follow Struwe's notation. Let $z = (x, t)$ denote points in $\mathbf{R}^m \times \mathbf{R}_+$. For a distinguished point $z_0 = (x_0, t_0)$, $R > 0$, let $\mathbf{B}_R(x_0) = \{x: |x - x_0| < R\}$ be a Euclidean ball centered at x_0. Also let $T_R(t_0) = \{z = (x, t)| t_0 - 4R^2 < t < t_0 - R^2\}$ and $S_R(t_0) = \{z(x, t): |t = t_0 - R^2\}$. Define the fundamental solution

$$G_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right), \qquad t < t_0.$$

In [3], Struwe proved that

$$\Sigma = \bigcap_{R>0} \left\{ z_0 \in \mathbf{R}^m \times \mathbf{R}_+ \,\middle|\, \liminf_{k\to\infty} \int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0}\, dx\, dt \geq \epsilon_0 \right\},$$

where ϵ_0 is the constant determined in Theorem 5.1 of [3]. Moreover, Σ is a closed set by Theorem 6.1 of [3].

Let

$$\Sigma_R^{t_0} = \left\{ x_0 \in \mathbf{R}^m \,\middle|\, \liminf_{k\to\infty} \int_{T_R(t_0)} |\nabla u_k|^2 G_{(x_0, t_0)}\, dx\, dt \geq \epsilon_0 \right\},$$

and let $\Sigma^{t_0} = \bigcap_{R>0} \Sigma_R^{t_0}$; then $\Sigma = \bigcup_{t_0>0} \Sigma^{t_0}$. For the theorem we will actually show that

$$\mathbf{H}^{m-2}(\Sigma^{t_0}) < C(t_0),$$

where $C(t_0)$ is a finite number depending only on the time t_0 (as well as the target manifold $\mathcal{N}$, the dimension m, and the energy bound E_0).

3. Proof of Theorem

Lemma 1 [3, (5.4′) and (5.4″)]. *For $\epsilon > 0$, one has on $T_R(t_0)$ the estimate*

$$G_{z_0}(x, t) \leq \begin{cases} R^m & \text{for all } x, \\ \epsilon G_{z_0 + (o, R^2)}(x, t) & \text{if } |x - x_0| > K(\epsilon) R, \end{cases}$$

where $K(\epsilon)$ depends only on ϵ and m.

Proof. For any (x, t) in $T_R(t_0)$,

$$G_{z_0} = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right) < \frac{1}{R^m}.$$

In the case $|x - x_0| > K(\epsilon)R$, we can estimate

$$\begin{aligned}\frac{G_{z_0}}{G_{z_0+(0,R^2)}} &= \frac{(t_0 - t + R^2)^{m/2}}{(t_0 - t)^{m/2}} \exp\left(\frac{|x - x_0|^2}{4(t_0 - t + R^2)} - \frac{|x - x_0|^2}{4(t_0 - t)}\right) \\ &\le 5^{m/2} \exp\left(-\frac{R^2|x - x_0|^2}{4(t_0 - t + R^2)(t_0 - t)}\right) \\ &\le 5^{m/2} \exp\left(-\frac{K^2(\epsilon)R^4}{4 \cdot 5R^2 \cdot 4R^2}\right) = 5^{m/2}\exp(-K^2(\epsilon)/80) < \epsilon\end{aligned}$$

for a suitable $K(\epsilon)$.

Lemma 2 [3, Lemma 3.2]. *Let* $u: \mathbf{R}^m \times [0, T] \to \mathcal{N}$ *be a regular solution to* (1.1) *with* $|\nabla u(x, t)| \le c < \infty$ *uniformly. Then for any* $z_0 = (x_0, t_0) \in \mathbf{R}^m \times (0, T)$ *the function*

$$\Phi_{z_0}(R; u) = \frac{1}{2}R^2 \int_{S_R(t_0)} |\nabla u|^2 G_{z_0}\, dx$$

is nondecreasing in R *for* $0 < R \le R_0 = \sqrt{t_0}$.

Lemma 3 [3, Proposition 3.3]. *Let* u *be as in Lemma* 2. *Then the function*

$$\Psi_{z_0}(R, u) = \int_{T_R(t_0)} |\nabla u|^2 G_{z_0}\, dx\, dt$$

is nondecreasing in R *for* $0 < R \le R_0$.

Note that Lemma 3 implies that if $R_1 < R_2$, then $\Sigma^{t_0}_{R_1} \subset \Sigma^{t_0}_{R_2}$.

For the proofs of Lemmas 2 and 3, see [3, Lemma 3.2 and Proposition 3.3].

Proof of the Theorem. By Lemma 1 we obtain

$$\begin{aligned}\int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0}\, dx\, dt &\le \int_{t_0-4R^2}^{t_0-R^2} \int_{\mathbf{B}_{KR}(x_0)} R^{-m} |\nabla u_k|^2\, dx\, dt \\ &\quad + \epsilon \int_{t_0-4R^2}^{t_0-R^2} \int_{|x-x_0|\ge K(\epsilon)R} |\nabla u_k|^2 G_{z_0+(0,R^2)}\, dx\, dt \\ &\le R^{-m} \int_{t_0-4R^2}^{t_0-R^2} \int_{\mathbf{B}_{KR}(x_0)} |\nabla u_k|^2\, dx\, dt \\ &\quad + \epsilon \int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0+(0,R^2)}\, dx\, dt.\end{aligned}$$

Now applying Lemma 2 to the last term yields

$$
\begin{aligned}
\epsilon \int_{T_R(t_0)} & |\nabla u_k|^2 G_{z_0+(0,R^2)}\,dx\,dt \\
&= \epsilon \int_{t_0-4R^2}^{t_0-R^2} 2(R^2+t_0-t)^{-1}\Phi_{z_0+(0,R^2)}(\sqrt{R^2+t_0-t}\,,\,u_k)\,dt \\
&\le \epsilon \int_{t_0-4R^2}^{t_0-R^2} 2(R^2+t_0-t)^{-1}\Phi_{z_0+(0,R^2)}(\sqrt{t_0+R^2}\,,\,u_k)\,dt \\
&\le \epsilon (t_0+R^2)^{1-m/2}\left\{\int_{\mathbf{R}^m}|\nabla u_k|^2\,dx|_{t=0}\right\}\int_{t_0-4R^2}^{t_0-R^2}(R^2+t_0-t)^{-1}\,dt \\
&\le \epsilon (t_0+R^2)^{1-m/2}E_0\log 5/2 \le \epsilon t_0^{1-m/2}E_0 \le \frac{1}{2}\epsilon_0
\end{aligned}
$$

for ϵ sufficiently small depending on E_0, m, and t_0. So we have

$$
\int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0}\,dx\,dt \le \frac{1}{2}\epsilon_0 + R^{-m}\int_{t_0-4R^2}^{t_0-R^2}\int_{\mathbf{B}_{KR}(x_0)}|\nabla u_k|^2\,dx\,dt.
$$

Now K depends on ϵ_0, E_0, m, $\mathcal{N}$, and t_0.

If $x_0 \in \Sigma_R^{t_0}$, then

$$
\begin{aligned}
\epsilon_0 &\le \liminf_{k\to\infty}\int_{T_R(t_0)}|\nabla u_k|^2 G_{z_0}\,dx\,dt \\
&\le \frac{1}{2}\epsilon_0 + \liminf_{k\to\infty} R^{-m}\int_{t_0-4R^2}^{t_0-R^2}\int_{\mathbf{B}_{KR}(x_0)}|\nabla u_k|^2\,dx\,dt,
\end{aligned}
$$

and therefore

$$
R^m \le \frac{2}{\epsilon_0}\liminf_{k\to\infty}\int_{t_0-4R^2}^{t_0R^2}\int_{\mathbf{B}_{KR}(x_0)}|\nabla u_k|^2\,dx\,dt.
$$

Observe that the family $\mathscr{F} = \{\mathbf{B}_{KR}(x_0) | (x_0 \in \Sigma_R^{t_0}\}$ covers $\Sigma_R^{t_0} \cap F$ for compact $F \subset \mathbf{R}^m$, so there is a finite subfamily $\mathscr{F}' = \{\mathbf{B}_{KR}(x_j)\}$ such that any two balls in $\mathscr{F}'$ are disjoint and that $\{\mathbf{B}_{5KR}(x_j)\}$ covers $\Sigma_R^{t_0} \cap F$.

Thus,

$$\begin{aligned}\Sigma_j(5KR)^m &= (5K)^m \Sigma_j R^m \\ &\le (5K)^m \Sigma_j \liminf_{k\to\infty} \frac{2}{\epsilon_0} \int_{t_0-4R^2}^{t_0-R^2} \int_{\mathbf{B}_{KR}(x_j)} |\nabla u_k|^2 \, dx \, dt \\ &\le C(5K)^m \liminf_{k\to\infty} \Sigma_j \int_{t_0-4R^2}^{t_0-R^2} \int_{\mathbf{B}_{KR}(x_j)} |\nabla u_k|^2 \, dx \, dt \\ &\le C(5K)^m \liminf_{k\to\infty} \int_{t_0-4R^2}^{t_0-R^2} \int_{\mathbf{R}^m} |\nabla u_k|^2 \, dx \, dt \\ &\le C(5K)^m \int_{t_0-4R^2}^{t_0-R^2} E_0 \, dt \le C(5K)^m E_0 \cdot 3R^2,\end{aligned}$$

and therefore

$$\sum_j (5KR)^{m-2} \le C(5K)^{m-2} E_0.$$

Hence,

$$\begin{aligned}&\mathbf{H}^{m-2}(\Sigma^{t_0} \cap F) \\ &\quad = \lim_{r\to 0} \left\{ \inf \left\{ \omega_{m-2} \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam} A_i}{2} \right)^{m-2} \middle| \Sigma^{t_0} \cap F \right.\right. \\ &\qquad\qquad\qquad \left.\left. \subset \bigcup_{i=1}^{\infty} A_i, \ \operatorname{diam} A_i \le r \right\}\right\} \\ &\quad \le \lim_{R\to 0} \omega_{m-2} \sum_j (5KR)^{m-2} \le C(t_0),\end{aligned}$$

where $C(t_0) = \omega_{m-2} C(5K)^{m-2} E_0$, and ω_{m-2} is the volume of the unit ball in $\mathbf{R}^{m-2}$.

Since F is arbitrary, we obtain the desired result:

$$\mathbf{H}^{m-2}(\Sigma^{t_0}) \le C(t_0). \qquad \text{q.e.d.}$$

Examining the specific dependence of $C(t_0)$ on t_0 as well as $\mathcal{N}$, m, and E_0, we see that

$$C(t_0) \le C_1(C_2 - \log t_0)^{(m-2)/2},$$

where C_1 and C_2 are positive constants depending only on $\mathcal{N}$, m, and E_0. Struwe [3] has observed that Σ^{t_0} is actually empty for $t_0 > T_0$, where T_0 is a positive constant depending only on $\mathcal{N}$, m, and E_0.

As in [1], the above estimate continues to hold. We then conclude:

For any smooth $u_0 : \mathcal{M} \to \mathcal{N}$, *there exists a global weak solution* $u : \mathcal{M} \times \mathbf{R}_+ \to \mathcal{N}$ *of the evolution problem for harmonic maps* (1.1). u *is regular off a singular closed set* $\Sigma \subset \mathcal{M} \times \mathbf{R}_+$, *and* $\Sigma \cap (\mathcal{M} \times \{t_0\})$ *has finite* $(m-2)$*-dimensional Hausdorff measure.*

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